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REALIZATION THEORY FOR MULTIVARIATE GAUSSIAN PROCESSES II: STATE SPACE THEORY REVISITED AND DYNAMICAL REPRESENTATIONS OF FINITE DIMENSIONAL STATE SPACES.*

Anders Lindquist† and Giorgio Picci‡

ABSTRACT: The purpose of this paper is twofold. First, some of the results of Part I are generalized and clarified, reformulated in a mathematical framework which we now perceive as more natural and conceptually sounder, the basic concept being perpendicularly intersecting subspaces. By relating minimality of stochastic realizations to the factorization of certain Hankel operators, the connections to deterministic realization theory is further clarified. Furthermore, spectral domain criteria for minimality, observability and constructibility are presented. Secondly, stochastic differential equation representations are presented for all state spaces which are finite dimensional. The infinite dimensional case will be the topic of Part III of this sequence of papers.

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1. INTRODUCTION

This is Part II in a sequence of papers, the first one being [1]. We have tried to make the presentation essentially self-contained by restating certain results without proofs, but we shall frequently refer the reader to Part I for details. Part I also contains a more detailed introduction to the problem under study and an historical account of it, as well as a relevant list of references.

Given an m-dimensional stationary stochastic process $\{y(t);\ t \in R\}$, which is real, Gaussian, purely nondeterministic, mean square continuous and centered, the basic problem is to find representations of the type

$$y(t) = Cx(t) \tag{1.1}$$

where C is a linear time-invariant operator and $\{x(t);\ t \in R\}$ is a (generally infinite dimensional) vector Markov process, for which we can obtain a differential equation representation.

This problem can be given a mathematically more precise formulation in a Hilbert space setting. To this end, define H to be the Gaussian space generated by y, i.e., the linear hull of the random variables $\{y_k(t);\ t \in R,\ k = 1,2 \ldots, m\}$ closed in L_2 norm. The space H is a Hilbert space with inner product $<\xi,\eta> = E\{\xi\eta\}$, where $E\{\cdot\}$ denotes mathematical expectation. To describe the dynamical aspects of the problem we need to define the past and the future of H and to introduce a shift operator on H: Let the past space H^- and the future space H^+ be the closed linear hulls in H of $\{y_k(t);\ t \le 0,\ k = 1,2, \ldots, m\}$ and $\{y_k(t);\ t \ge 0,\ k = 1,2, \ldots, m\}$ respectively. Moreover, since y is stationary, there is a strongly continuous group $\{U_t;\ t \in R\}$ of unitary operators $H \to H$ such that $y_k(t) = U_{t-s}y_k(s)$ for all t,s and k[2]. Given three arbitrary subspaces A,B and X of H, we shall write $A \perp B$ when A and B are orthogonal and $A \perp B|X$ when they are conditionally orthogonal

given X, i.e., $\langle E^X\alpha, E^X\beta\rangle = \langle\alpha,\beta\rangle$ for all $\alpha \in A$ and $\beta \in B$, where E^X denotes orthogonal projection onto X. Finally we shall write $A \vee B$ to denote the closed linear hull of A and B.

The problem at hand can now be formulated as a geometric problem in the Hilbert space H: Find all subspaces X such that

(i) X is *Markovian*, i.e.

$$X^- \perp X^+ | X \tag{1.2}$$

where $X^- := \bigvee_{t \leq 0} (U_t X)$ and $X^+ := \bigvee_{t \geq 0} (U_t X)$.

(ii) $y_k(0) \in X$ for $k = 1, 2, \ldots, m$

(iii) X is *minimal*, i.e., there is no proper subspace of X satisfying (i) and (ii).

Each solution X of this problem gives rise to a representation of type (1.1). In fact, a vector process $\{x(t);\ t \in R\}$ taking values in some suitable function space can be constructed by appropriately choosing a basis in X and applying the shift U_t to its components. (Note that, since y is mean-square continuous, H is separable.) In view of (i), this process must be Markov. Condition (ii) then yields a representation (1.1). Finally, (iii) insures that the *state space* X is as small as possible. In Section 7 we shall describe how this choice of basis is made in the finite dimensional case ($\dim X < \infty$), postponing the infinite dimensional case to a subsequent paper, Part III. (The geometric theory of Sections 2-6 does not, however, require any restriction of the dimension of X.)

This paper constitutes not merely a sequel of Part I; it contains many extensions, generalizations and other improvements of the results presented in Part I. We have found that the geometric theory is most easily understood and explained in terms of *perpendicularly intersecting subspaces*, and we devote Sections 2 and 3 to this reformulation. In Section 4, we show how the

concepts of deterministic realization theory [3,4] can be used to provide further insight into the stochastic theory, and Section 5 and 6 contain extensions of the spectral domain theory presented in [1]. Section 7, finally, is devoted to dynamical representations of finite dimensional state spaces.

2. PERPENDICULARLY INTERSECTING SUBSPACES

Let A and B be two (closed) subspaces of the Hilbert space H. We shall say that the subspace $X \subset H$ is an *(A,B)-splitting subspace* if

$$A \perp B | X . \tag{2.1}$$

If there is no *proper* subspace of X which also satisfies (2.1), X is said to be *minimal*. The purpose of this section is to determine under what conditions on A and B there is only *one* minimal (A,B)-splitting subspace.

LEMMA 2.1 *There is one and only one minimal (A,B)-splitting subspace contained in* A (*in* B), *namely* $X = \bar{E}^A B$ ($X = \bar{E}^B A$). (The bar over the E denotes closure.)

Proof. The useful decomposition formula

$$A = \bar{E}^A B \oplus (A \cap B^\perp) \tag{2.2}$$

implies that $\bar{E}^A B = A \ominus (A \cap B^\perp)$ and that $A \ominus \bar{E}^A B \perp B$, the latter of which is equivalent to (2.1) with $X = \bar{E}^A B$. On the other hand, any (A,B)-splitting subspace X containing A satisfies $A \ominus X \perp B$, i.e., $A \ominus X \subset A \cap B^\perp$, or equivalently $X \supset A \ominus (A \cap B^\perp)$, i.e., $X \supset \bar{E}^B A$. □

LEMMA 2.2 *All* (A,B)*-splitting subspaces contain* $A \cap B$.

Proof. Let $\eta \in A \cap B$. Then $\eta \perp \eta | X$, i.e., $\eta \in X$. □

Consequently, if there is only one (A,B)-splitting subspace, we have $\bar{E}^A B = \bar{E}^B A = A \cap B$. Conversely, if

$$\bar{E}^A B = A \cap B, \tag{2.3}$$

and it must be the only minimal (A,B)-splitting subspace, since all such subspaces contain $A \cap B$ (Lemma 2.2). We shall say that A and B *intersect perpendicularly* if one of the two equivalent conditions (2.3) or (2.4) holds.

PROPOSITION 2.1. *There is a unique minimal* (A,B)*-splitting subspace if and only if* A *and* B *intersect perpendicularly and it is given by (2.3) and (2.4).*

The following proposition provides an alternative characterization of perpendicular intersection of subspaces in the case that they span the whole space.

PROPOSITION 2.2. *Let* $A \vee B = H$. *Then* A *and* B *intersect perpendicularly if and only if* $B^{\perp} \subset A$, *or equivalently,* $A^{\perp} \subset B$. (Here $A^{\perp}$ denotes the orthogonal complement of A in H.)

Proof. (if): Assume that $B^{\perp} \subset A$ holds. Set $X = \bar{E}^{A}B$. Then (2.2) yields $A = X \oplus B^{\perp}$, from which it follows that $X \subset B$ and that $A \cap B = X$.

(only if): Set $X = \bar{E}^{A}B$ and assume that $X = A \cap B$. Then it follows from (2.2) that $(A \ominus X) \perp B$. Therefore, since $A \vee B = H$ and $B \supset X$, $B^{\perp} \subset A$. □

In the sequel, we shall need the following simple observation, the proof of which is trivial.

LEMMA 2.3. *Let* A, B, A' *and* B' *be four subspaces such that* $A \subset A'$ *and* $B \subset B'$. *Then any* (A',B')*-splitting subspace is also an* (A,B)*-splitting subspace.*

3. THE GEOMETRY OF SPLITTING SUBSPACES REVISITED

Our basic problem is to determine the set of all minimal (H^{-},H^{+})-splitting subspaces, where H^{-} and H^{+} are the past and future spaces of the

given stochastic process $\{y(t);\ t \in R\}$; we shall usually drop the prefix (H^-,H^+), plain "splitting subspace" referring to the pair (H^-,H^+). If H^- and H^+ intersect perpendicularly there is only one minimal splitting subspace, namely the *present space*

$$H^{\circ} = H^- \cap H^+ \tag{3.1}$$

(Proposition 2.1). When y has a rational spectral density, this corresponds to the case where y can be realized by a purely autoregressive scheme. However, in general this is not the case, and there is a whole family of minimal splitting subspaces, two of which are the forward and backward predictor spaces $X_- := \bar{E}^{H^-} H^+$ and $X_+ := \bar{E}^{H^+} H^-$ respectively (Lemma 2.1). Defining $N^- := H^- \ominus X_-$ and $N^+ := H^+ \ominus X_+$ we obtain the orthogonal decomposition

$$H = N^- \oplus H^{\square} \oplus N^+, \tag{3.2}$$

where H is the Gaussian space of the process y and $H^{\square}$ is the *frame space* $H^{\square} := X_- \vee X_+$. In Part I, we showed that all minimal splitting subspaces X satisfy

$$H^{\circ} \subset X \subset H^{\square}, \tag{3.3}$$

i.e., the frame space is the closed linear hull of all minimal splitting subspaces, hence containing all pertinent information about y. We shall call N^- and N^+ the *junk spaces* since all information in them may be discarded. By applying (2.2), it is immediately seen that $N^- = H^- \cap (H^+)^{\perp}$ and $N^+ = H^+ \cap (H^-)^{\perp}$. In the special case that H^- and H^+ intersect perpendicularly $H^{\square}$ and H° coincide.

The following theorem expresses the fact that any (minimal or nonminimal) splitting subspace can be regarded as a *minimal* splitting subspace if the past and future spaces are extended so that they intersect perpendicularly.

THEOREM 3.1. *The space* $X \subset H$ *is a splitting subspace if and only if it is the (unique) minimal* $(S,\bar{S})$*-splitting subspace*

$$X = S \cap \bar{S} \tag{3.4}$$

for some perpendicularly intersecting subspaces S *and* $\bar{S}$ *such that* $S \supset H^-$ *and* $\bar{S} \supset H^+$. *The correspondence* $X \leftrightarrow (S,\bar{S})$ *is one-one, the pair* $(S,\bar{S})$ *being uniquely determined by the relations* $S = H^- \vee X$ *and* $\bar{S} = H^+ \vee X$.

COROLLARY 3.1. *Representation (3.4) may also be written in any of the following four equivalent ways:*

$$X = S \ominus \bar{S}^{\perp} \tag{3.5}$$

$$X = \bar{S} \ominus S^{\perp} \tag{3.6}$$

$$X = \bar{E}^{S}\bar{S} \tag{3.7}$$

$$X = \bar{E}^{\bar{S}}S. \tag{3.8}$$

Proof. Theorem 3.1 in Part I states that X is a splitting subspace if and only if (3.5) or (3.6) holds for some $S \supset H^-$ and $\bar{S} \supset H^+$, and that X and $(S,\bar{S})$ are related as in the last sentence of the present theorem. But then $S^{\perp} \subset \bar{S}$, and therefore S and $\bar{S}$ intersect perpendicularly (Proposition 2.2) and X is given by (3.4) (Proposition 2.1). Relations (3.7) and (3.8) follow by comparing (3.5) and (3.6) to (2.2). □

A subspace containing the past space H^- will be called an *extended past space* and one containing the future H^+ an *extended future space*. We shall say that an extended past space S (extended future space $\bar{S}$) is *minimal* if there is a minimal splitting subspace X such that $S = H^- \vee X$ ($\bar{S} = H^+ \vee X$). This definition coincides with that in [5] for regular Markovian splitting subspaces (to be defined shortly).

PROPOSITION 3.1. *An extended past space* S *is minimal if and only if*

$$S \subset (N^+)^\perp \tag{3.9}$$

and an extended future space $\bar{S}$ *is minimal if and only if*

$$\bar{S} \subset (N^-)^\perp. \tag{3.10}$$

Proof. (if): In the first case, choose $\bar{S} = S^\perp \vee H^+$ and in the second case, $S = \bar{S}^\perp \vee H^-$. Then it follows from Theorem 3.1 in Part I that $X := S \cap \bar{S}$ is the required X.

(only if): Since a minimal splitting subspace X is contained in $H^\square$, we have $S = H^- \vee X \subset (N^+)^\perp$ and $\bar{S} = H^+ \vee X \subset (N^-)^\perp$. □

The reason for this definition of minimality will become clear in Section 5. Note that an arbitrary pair $(S,\bar{S})$ of perpendicularly intersecting minimal extended past and future spaces will not necessarily define a *minimal* splitting subspace X, although any such X will be contained in the frame space $H^\square$. For X to be minimal, the pairing of S and $\bar{S}$ must be done in a more precise manner. This leads to the concepts of observability and constructibility as defined by Ruckebusch.

A splitting subspace is said to be *observable* if $\bar{E}^X H^+ = X$ and *constructible* if $\bar{E}^X H^- = X$. In Part I, we showed that $X = S \cap \bar{S}$ is observable if and only if

$$\bar{S} = S^\perp \vee H^+ \tag{3.11}$$

and constructible if and only if

$$S = \bar{S}^\perp \vee H^-. \tag{3.12}$$

PROPOSITION 3.2. *Let* X *be a splitting subspace, and let* S *and* $\bar{S}$ *be the corresponding extended past and future spaces. Then the following conditions are equivalent:*

(i) X *is minimal*

(ii) X *is observable and constructible*

(iii) X *is observable and* S *is minimal*

(iv) X *is constructible and* $\bar{S}$ *is minimal*

For a proof of the equivalence of (i) and (ii), see [6] or Proposition 4.2 below. Keeping Proposition 3.1 in mind, the rest follows from Theorem 3.1 of Part I. Note that minimality of X imposes a definite pairing of S and $\bar{S}$. For example, X_- corresponds to $S_- := H_-$ and $\bar{S}_- := (N^-)^{\perp}$ and X_+ to $S_+ := (N^+)^{\perp}$ and $\bar{S}_+ := H^+$.

Among all minimal splitting subspaces X we shall eventually only be able to use those which are *Markovian*, i.e., those which satisfy condition (1.2). As pointed out in Part I (the original reference is [6]), X is Markovian if and only if the corresponding extended spaces S and $\bar{S}$ satisfy the following two conditions:

$$U_t S \subset S \quad \text{for all} \quad t \le 0 \tag{3.13}$$

$$U_t \bar{S} \subset \bar{S} \quad \text{for all} \quad t \ge 0. \tag{3.14}$$

i.e., S must be left invariant and $\bar{S}$ right invariant (under the shift U_t). Note that, if X is observable, we need only impose condition (3.13); then, in view of (3.11), (3.14) will follow automatically. In the same way, only (3.14) is needed when X is constructible. It is easy to see that for a Markovian splitting subspace $S = X^-$ and $\bar{S} = X^+$, i.e., X^- and X^+ intersect perpendicularly. In general, S and $\bar{S}$ are given by $S = S_1 \oplus S_2$ and $\bar{S} = \bar{S}_1 \oplus \bar{S}_2$ where S_2 and $\bar{S}_2$ are *doubly invariant*, i.e., they satisfy both (3.13) and (3.14), whereas S_1 and $\bar{S}_1$ are only simply invariant. Then $\cap_{t \in R} U_t S = S_2$ and $\cap_{t \in R} U_t \bar{S} = \bar{S}_2$, i.e., S_2 and $\bar{S}_2$ are the purely deterministic parts of S and $\bar{S}$ respectively. If $S_2 = 0$ ($\bar{S}_2 = 0$), we shall say

that S $(\bar{S})$ is *purely nondeterministic*. A splitting subspace for which both extended spaces S and $\bar{S}$ are purely nondeterministic will be called *regular*. In view of Lemma 2.1 of Part I and the fact that $S \subset \bar{S}$, both S and $\bar{S}$ are full range whenever X is regular. The usefulness of this concept is that the class of regular Markovian splitting subspaces is in one-one correspondence with a certain class of pairs $(W,\bar{W})$ of full-rank factors of y, W being stable and $\bar{W}$ strictly unstable. We shall return to this in Section 5.

To insure that all minimal splitting subspaces are regular, we need to introduce the concept of noncyclicity. We say that the given process y is *noncyclic* if it has nontrivial junk spaces, i.e., $N^- \neq 0$ and $N^+ \neq 0$, and *strictly noncyclic* if both N^- and N^+ are full range.

PROPOSITION 3.3 *Let* y *be strictly noncyclic. Then all minimal splitting subspaces are regular.*

Proof. If N^- and N^+ are full range, $(N^-)^\perp$ and $(N^+)^\perp$ are purely nondeterministic (Part I; Lemma 2.1). Hence, the proposition follows from (3.9) and (3.10). □

4. HANKEL OPERATORS AND MINIMALITY OF SPLITTING SUBSPACES

Given an arbitrary splitting subspace X, there is a unique pair $(S,\bar{S})$ of perpendicularly intersecting subspaces such that $S \supset H^-$, $\bar{S} \supset H^+$ and $X = S \cap \bar{S}$ (Theorem 3.1). We shall try to gain some further insight into the conditions under which the state space X is minimal by applying the basic concepts of deterministic realization theory [3,4]. Let $G\colon \bar{S} \to S$ and $G^*\colon \bar{S} \to S$ be the *Hankel operators* $G = E^S|_{\bar{S}}$ and $G^* = E^{\bar{S}}|_S$, where $A|_B$ denotes the restriction of the operator A to the domain B. Furthermore, let $\bar{R}(A)$ denote the closure of the range and $N(A)$ the null space of A.

PROPOSITION 4.1. *The operators* G *and* G* *are adjoints. Moreover,* $N(G) = S^{\perp}$, $N(G^*) = \bar{S}^{\perp}$ *and*

$$\bar{R}(G) = \bar{R}(G^*) = X, \tag{4.1}$$

where $X = S \cap \bar{S}$.

Proof. To see that the first statement is true, note that for all $s \in S$ and $\bar{s} \in \bar{S}$

$$\langle G\bar{s}, s\rangle_S = \langle \bar{s}, s\rangle = \langle \bar{s}, G^*s\rangle_{\bar{S}},$$

where $\langle\cdot,\cdot\rangle_S$ is the inner product restricted to S. Relations (4.1) follow from (3.7) and (3.8), and, since $S^{\perp} \subset \bar{S}$ (Proposition 2.2), $N(G) = S^{\perp}$. In the same way, it is seen that $N(G^*) = \bar{S}^{\perp}$. □

By Corollary 3.1, $X = S \ominus \bar{S}^{\perp} = \bar{S} \ominus S^{\perp}$. Consequently, X is isomorphic to the quotient spaces $S/\bar{S}^{\perp}$ and $\bar{S}/S^{\perp}$, i.e.,

$$X \cong \bar{S}/N(G) \quad \text{and} \quad X \cong S/N(G^*) \tag{4.2}$$

(Proposition 4.1). Actually, by choosing representations in the equivalence classes properly, we may identify X directly with the quotient spaces in (4.2).

The formulas (4.1) show that the factorizations of G and G* through X described by the commutative diagrams

$$\begin{tikzcd} \bar{S} \arrow[rr, "G"] \arrow[dr, "E^X|_{\bar{S}}"'] & & S \\ & X \arrow[ur, "J"'] & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[rr, "G^*"] \arrow[dr, "E^X|_{S}"'] & & \bar{S} \\ & X \arrow[ur, "\bar{J}"'] & \end{tikzcd} \tag{4.3}$$

(where $J: X \to S$ and $\bar{J}: X \to \bar{S}$ are the insertion maps $Jx = \bar{J}x = x$) are *canonical* in the sense that $E^X|_{\bar{S}}$ $(E^X|_S)$ maps *onto* a dense subset of X and J $(\bar{J})$ is *one-one*. This canonical property, as well as (4.2), illustrates the fact that X is a minimal $(S,\bar{S})$-splitting subspace. (Cf[3; p.259].)

We shall now relate this approach to the concepts of observability and constructibility defined in Section 3 by investigating under what conditions X is a *minimal* splitting subspace [with respect to (H^-, H^+)]. The following lemma provides us with two splitting subspaces which are potentially smaller.

LEMMA 4.1. *Consider the restricted Hankel operators* $G^+ := G|_{H^+}$ *and* $G^- := G^*|_{H^-}$. *Then* $\bar{R}(G^+)$ *and* $\bar{R}(G^-)$ *are splitting subspaces contained in* X.

Proof. It suffices to prove the statement concerning G^+; then the one about G^- follows by symmetry. Trivially $R(G^+) \subset R(G)$, and hence $\bar{R}(G^+) \subset X$ (Proposition 4.1). Obviously, the predictor space $\bar{R}(G^+) = \bar{E}^S H^+$ is an (S, H^+)-splitting subspace, and hence also a splitting subspace (Lemma 2.3). □

It is therefore natural to consider the factorizations of G^+ and G^- through X induced by those in (4.3), i.e.,

$$\begin{tikzcd} H^+ \arrow[rr, "G^+"] \arrow[dr, "O"'] & & S \\ & X \arrow[ur, "J"'] & \end{tikzcd} \qquad \begin{tikzcd} H^- \arrow[rr, "G^-"] \arrow[dr, "C"'] & & \bar{S} \\ & X \arrow[ur, "\bar{J}"'] & \end{tikzcd} \tag{4.4}$$

where $O := E^X|_{H^+}$ is the *observability operator* and $C := E^X|_{H^-}$ the *constructibility operator*. The first factorization is canonical if and only if O maps onto a dense subset of X, i.e., X is observable, in which case $X = \bar{R}(G^+)$. The same statement holds for the second factorization (4.4) if we exchange O for C, G^+ for G^- and observable for constructible. Consequently, minimality of X is equivalent to both factorizations being canonical (Proposition 3.2). Note that, unlike the situation in deterministic realization theory [3,4], *two* factorizations are needed. In this respect, constructibility is a form of dual observability and does not correspond to reachability. The "reachability operators" J and its dual $\bar{J}$ are always one-one and play

no important role here. (Since in effect we are working with spaces of functionals of inputs and outputs, both diagrams are dualized as compared to the deterministic setting in the sense that all arrows and the order between the observability and reachability operators have been reversed.)

The previous discussion suggests that a splitting subspace X is observable if and only if, for a given S, $\bar{S}$ is as small as possible, and that, dually, the same holds for constructibility exchanging S for $\bar{S}$ and vice versa. To clarify this point, first consider the following lemma.

LEMMA 4.2. *Let* X *and* X' *be two splitting subspaces and let* $(S,\bar{S})$ *and* $(S',\bar{S}')$ *be the corresponding extended past and future spaces. Then* $X \subset X'$ *if and only if* $S \subset S'$ *and* $\bar{S} \subset \bar{S}'$.

Proof. In view of (3.4), the (if)-part is trivial. It remains to prove (only if). Since $S = H^- \vee X$ (Theorem 3.1), $X' \subset X$ implies that $S' \subset S$. The rest follows by symmetry. □

Consequently, in order to obtain a minimal $X = S \cap \bar{S}$, we must reduce S and $\bar{S}$ as much as possible. Now, keeping S fixed, it is easy to see that the smallest $\bar{S}$ which both contains H^+ and intersects S perpendicularly is $\bar{S} = H^+ \vee S^\perp$, i.e., exactly the condition (3.11) for observability. Likewise, keeping $\bar{S}$ fixed, $S = H^- \vee \bar{S}^\perp$ is the smallest subspace containing H^- and intersecting $\bar{S}$ perpendicularly, yielding the constructibility condition (3.12). We shall now discuss some consequences of these observations.

First, this provides us with an alternative proof of the equivalence of conditions (i) and (ii) in Proposition 3.2, which we restate here in a somewhat stronger form.

PROPOSITION 4.2. *Let* $S \supset H^-$ *and* $\bar{S} \supset H^+$ *be two arbitrary subspaces, and set* $X = S \cap \bar{S}$. *Then* X *is a minimal splitting subspace if and only if both conditions (3.11) and (3.12) hold.*

Proof. (if): Assume that (3.11) and (3.12) hold. The $S^\perp \subset \bar{S}$, i.e., S and $\bar{S}$ intersect perpendicularly (Proposition 2.2). Hence, X is a splitting subspace (Theorem 3.1). It remains to show that X is minimal. In order to reduce X either S or $\bar{S}$ or both must be reduced while neither can be enlarged (Lemma 4.2). But this is inconsistent with (3.11) and (3.12), since S and $\bar{S}$ will not be reduced. Hence X is minimal.

(only if): Suppose that X is a minimal splitting subspace. Then the pair $(S,\bar{S})$ is uniquely determined; we must have $S = H^- \vee X$ and $\bar{S} = H^+ \vee X$ as in Theorem 3.1. In fact, under the given conditions, neither S nor $\bar{S}$ can be smaller, and enlarging (say) S (as given by Theorem 3.1), to $S \oplus Z$ will yield $(S \oplus Z) \cap \bar{S} = X \oplus Z \neq X$, since $Z \subset S^\perp \subset \bar{S}$ (Proposition 2.2). Now assume that (say) (3.11) does not hold. Since $\bar{S}$ contains H^+ and intersects S perpendicularly, $\bar{S} \supset H^+ \vee S^\perp$. Then, by the discussion above, $\bar{S}$ can be reduced to $H^+ \vee S^\perp$. Since $\bar{S} = H^+ \vee X$, this would reduced X too, contradicting minimality. □

Secondly, given any splitting subspace, the above procedure defines an algorithm by which a minimal splitting subspace can be constructed.

PROPOSITION 4.3. *Let* X' *be an arbitrary splitting subspace and define* $\bar{S} := H^+ \vee (H^- \vee X')^\perp$ *and* $S := H^- \vee \bar{S}^\perp$. *Set* $X = S \cap \bar{S}$. *Then* X *is a minimal splitting subspace. If* X' *is Markovian, then so is* X.

Proof. (i) To show that X is a minimal splitting subspace, it only remains to prove that $\bar{S} = H^+ \vee S^\perp$ (Proposition 4.2). The given condition $S = H^- \vee \bar{S}^\perp$ implies that $S^\perp \subset \bar{S}$, and consequently $H^+ \vee S^\perp \subset \bar{S}$.

Set $S' = H^- \vee X'$. Then, since $(S')^\perp \subset \bar{S}$, S' contains H^- and intersects $\bar{S}$ perpendicularly. But S is the smallest subspace with this property. Hence, $S \subset S'$, i.e., $(S')^\perp \subset S^\perp$, which implies that $\bar{S} \subset H^+ \vee S^\perp$. Hence, $\bar{S} = H^+ \vee S^\perp$ as required.

(ii) To prove that X is Markovian, we must show that (3.13) and (3.14) hold. Since X' is Markovian, S' satisfies (3.13). Then $(S')^\perp$ satisfies (3.14) and, since the same is true for H^+, $\bar{S}$ also satisfies (3.14). Now, by a symmetric argument, it is seen that $S := H^- \vee \bar{S}^\perp$ satisfies (3.13). □

5. SPECTRAL REPRESENTATION OF REGULAR MARKOVIAN SPLITTING SUBSPACES

Since the given process y is stationary, mean-square continuous and purely nondeterminstic, it has the spectral representation

$$y(t) = \int e^{st}\, d\hat{y}(s) \tag{5.1}$$

where integration is over the imaginary axis and $d\hat{y}$ is an orthogonal stochastic vector measure such that

$$E\{d\hat{y}(i\omega)d\hat{y}(i\omega)^*\} = \Phi(i\omega)d\omega, \tag{5.2}$$

Φ being the mxm spectral density. Assume that Φ has rank $p \le m$. Then a *full-rank spectral factor* is any mxp-matrix solution of

$$W(s)W(-s)' = \Phi(s) \tag{5.3}$$

such that rank $W = p$. To any such spectral factor, we may associate a p-dimensional Wiener process

$$u(t) = \int \frac{e^{st}-1}{s} d\hat{u}(s) \;\; ; \; d\hat{u} = W^{-L}d\hat{y}, \tag{5.4}$$

which spans the whole space H (see Part I). (W^{-L} is the left inverse of W.) Let $\mathcal{U}$ denote the class of all such Wiener processes. Let $H^-(du)$ and $H^+(du)$ be the closed linear hulls in H of respectively the past and the future of

the process u. We shall say that W is *stable* if its rows belong to the Hardy space $\mathcal{H}_2^+$ of p-dimensional functions f which are analytic in the open right half-plane and such that $\int \|f(\sigma + i\omega)\|^2 d\omega$ is uniformly bounded for all $\sigma > 0$, and *strictly unstable* if its rows belong to the conjugate Hardy space $\mathcal{H}_2^-$. Let $\mathcal{U}^+$ and $\mathcal{U}^-$ be the subclasses of $\mathcal{U}$ corresponding to stable and strictly stable spectral factors respectively.

LEMMA 5.1. *There is a one-one correspondence between stable full-rank spectral factors* W *(determined modulo multiplication with a constant unitary matrix) and left invariant* [*i.e., satisfying (3.13)*] *and purely nondeterministic subspaces* $S \supset H^-$. *The subspace* S *is related to* W *by*

$$S = H^-(du) \tag{5.5}$$

where $u \in \mathcal{U}^+$ *is the Wiener process corresponding to* W.

LEMMA 5.2. *There is a one-one correspondence between strictly unstable full-rank spectral factors* $\overline{W}$ *and right invariant,* [*i.e., satisfying (3.14)*] *and purely nondeterministic subspaces* $\bar{S} \supset H^+$. *The subspace* $\bar{S}$ *is related to* $\overline{W}$ *by*

$$\bar{S} = H^+(d\bar{u}) \tag{5.6}$$

where $u \in \mathcal{U}^-$ *is the Wiener process corresponding to* $\overline{W}$.

The proofs of these lemmas can be found in Part I, where somewhat differently stated but equivalent results are given.

It was shown in Part I that, to each $u \in \mathcal{U}$,

$$\int f\, du \overset{Q_u}{\longmapsto} f \tag{5.7}$$

defines an isomorphism between H and the space $L_2^p(I)$ of all p-dimensional row vector functions which are square-integrable on the imaginary axis.

By the Paley-Wiener Theorem [7,8,11], the isomorphic image of $H^-(du)$ under Q_u is precisely the Hardy space H_2^+.

Now, in view of Theorem 3.1 and Lemmas 5.1 and 5.2, X is a regular Markovian splitting subspace if and only if

$$X = H^-(du) \cap H^+(d\bar{u}) \tag{5.8}$$

for some $u \in U^+$ and $\bar{u} \in U^-$ such that $H^-(du)$ and $H^+(d\bar{u})$ intersect perpendicularly.

LEMMA 5.3. *Let* $u \in U^+$ *and* $u \in U^-$, *and let* W *and* $\overline{W}$ *be the corresponding spectral factors. Set*

$$K = \overline{W}^{-L}W. \tag{5.9}$$

Then $S := H^-(du)$ *and* $\bar{S} := H^+(d\bar{u})$ *intersect perpendicularly if and only if* K *is an inner function.* (See Part I for definition.)

Proof. The subspaces S and $\bar{S}$ intersect perpendicularly if and only if $\bar{S}^\perp \subset S$ (Proposition 2.2), i.e., $H^-(d\bar{u}) \subset H^-(du)$, the isomorphic image of which (under Q_u) is $H_2^+K \subset H_2^+$; to see this use Lemma 4.3 of Part I. Since $\bar{S}^\perp$ is full range and left invariant, Lemma 4.1 of Part I can be used to show that $Q_u\bar{S}^\perp = H_2^+K$ is a full-range invariant subspace of $L_2^p(I)$ [7]. Hence, in view of the Beurling-Lax-Helson Theorem [7], $H_2^+K \subset H_2^+$ if and only if K is inner. □

By Corollary 3.1, (5.8) can be written $X = H^-(du) \ominus H^-(d\bar{u})$, the isomorphic image of which (under Q_u) is $Q_uX = H_2^+ \ominus (H_2^+K)$. Consequently

$$X = \int (H_2^+K)^\perp d\hat{u}, \tag{5.10}$$

where $u \in U^+$ is the Wiener process corresponding to W, and the superscript $^\perp$ denotes orthogonal complement in H_2^+. We collect these observations in the following theorem, which is a slight generalization of Theorem 4.1 in Part I.

THEOREM 5.1. *The subspace* X *is a regular Markovian splitting subspace if and only if (5.10) holds for some pair* $(W,\bar{W})$ *of full rank spectral factors such that* W *is stable,* $\bar{W}$ *is strictly unstable, and* $K := \bar{W}^{-L}W$ *is inner.*

In particular, if y is strictly noncyclic, all minimal Markovian splitting subspaces are given by Theorem 5.1 (Proposition 3.3).

6. SPECTRAL DOMAIN CRITERIA FOR OBSERVABILITY, CONSTRUCTIBILITY AND MINIMALITY

Theorem 5.1 provides us with a procedure to find all regular Markovian splitting subspaces: All possible pairs $(W,\bar{W})$ of full-rank spectral factors with W stable, $\bar{W}$ strictly unstable and $K = \bar{W}^{-L}W$ inner, inserted into (5.9) and (5.10), generate the whole family of such splitting subspaces. But how can we decide whether such a pair will provide an observable, or a constructible or a minimal splitting subspace? We need to translate the geometric criteria of Section 3 into spectral domain language.

To this end, first note that W is a stable full-rank spectral factor if and only if it can be written

$$W = W_-\theta, \tag{6.1}$$

where θ is an inner function (for H_2^+) and W_- is the unique outer spectral factor, corresponding to $S = S_- := H^-$. Similarly, $\bar{W}$ is a strictly unstable full-rank spectral factor if and only if it has the representation

$$\bar{W} = \bar{W}_+\bar{\theta}, \tag{6.2}$$

where $\bar{\theta}$ is conjugate inner (inner for H_2^-) and $\bar{W}_+$ is the unique conjugate outer spectral factor, which corresponds to $\bar{S} = \bar{S}_+ := H^+$.

PROPOSITION 6.1. *Let* X *be a regular Markovian splitting subspace, let (6.1) and (6.2) be the corresponding spectral factors, and let* K *be defined by (5.9). Then* X *is observable if and only if* K^{-1} *and* $\bar{\theta}$ *are right*

coprime and constructible if and only if K *and* θ *are right coprime.*

Proof. Let S and $\bar{S}$ be respectively the extended past and future spaces of X, and let u and $\bar{u}$ be the Wiener processes defined through the representations (5.5) and (5.6). Using Lemma 4.3 in Part I, it is seen that the isomorphic image of the constructibility condition (3.12) under the map Q_u is $H_2^+ = (H_2^+K) \vee (H_2^+\theta)$, which holds if and only if K and θ are right coprime. In the same way, using the map $Q_{\bar{u}}$, the observability condition (3.12) is seen to be equivalent to K^{-1} and $\bar{\theta}$ being right coprime. □

In order to apply conditions (iii) and (iv) of Proposition 3.2, we also need to characterize minimality of S and $\bar{S}$ in the spectral domain. We shall say that a stable (strictly unstable) full-rank spectral factor is *minimal* if it corresponds to a minimal extended past (future) space. Assume that y is strictly noncyclic. Then X_- and X_+ are regular splitting subspaces (Proposition 3.3), and consequently there are associated pairs $(W_-,\bar{W}_-)$ and $(W_+,\bar{W}_+)$ of full-rank spectral factors (Theorem 5.1). Then define K_- and K_+ to be the corresponding functions (5.9). We showed in Part I that in the scalar case, all minimal Markovian splitting subspaces have the same K, so that in particular $K_- = K_+$, but it is still unclear to us what happens in the general case. Moreover let θ_+ be the θ in (6.1) corresponding to W_+ and $\bar{\theta}_-$ the $\bar{\theta}$ in (6.2) corresponding to $\bar{W}$.

PROPOSITION 6.2. *Suppose that* y *is strictly noncyclic. Let* W *be a stable full-rank spectral factor, and define* $J = W^{-L}W_+$. *Then* J *is inner and the following conditions are equivalent:*

(i) W *is minimal*

(ii) θ *is a left inner divisor of* θ_+, *i.e., there is an inner function* ϕ *such that* $\theta\phi = \theta_+$.

(iii) J *and* K_+ *are right coprime.*

Proof. The equivalence between (i) and (ii) follows from Proposition 4.1 in Part I. The equivalence of (i) and (iii) follows from Theorem 5.1 in Part I. □

The fact that (i) and (ii) are equivalent was first proven in [5] in the scalar case. The dual version of Proposition 6.2 goes as follows.

PROPOSITION 6.3. *Suppose that* y *is strictly noncyclic. Let* $\bar{W}$ *be a strictly unstable full-rank spectral factor, and define* $\bar{J} = \bar{W}^{-L}\bar{W}_-$. *Then* $\bar{J}$ *is conjugate inner and the following conditions are equivalent:*

(i) $\bar{W}$ *is minimal*

(ii) $\bar{\theta}$ *is a left inner divisor of* $\bar{\theta}_-$

(iii) $\bar{J}$ *and* K_-^{-1} *are right coprime.*

7. STATE EQUATIONS FOR FINITE DIMENSIONAL MARKOVIAN SPLITTING SUBSPACES

If the matrix function K is rational the representation (5.10) is particularly simple to realize.

PROPOSITION 7.1. *Let* X *be a regular Markovian splitting subspace. Then* X *is finite dimensional if and only if* K, *defined by (5.9), is rational.*

Proof. As pointed out in Section 4, $X = \bar{R}(G)$, where $G = E^S|_{\bar{S}}$. Let u be the Wiener process associated to S through relation (5.5). Then, under Q_u, $S \simeq H_2^+$ and $\bar{S} \simeq H_2^- K$ (Part I: Lemma 4.3), and consequently, using the notations of Section 5 of Part I, G is isomorphic to the Hankel operator H_K. Since therefore $X \simeq \bar{R}(H_K)$, $\dim X < \infty$ if and only if K is rational [4; p. 395, Theorem 3.8]. (To prove the if-part, follow the lines of the last part of the proof of Lemma 7.1.) □

In general, there are no finite dimensional splitting subspaces. One very important exception, however, is the case when the spectral density of y is rational.

PROPOSITION 7.2. *Let the spectral density* Φ *be rational. Then all minimal splitting subspaces are finite dimensional.*

Proof. It is a well-known fact from Wiener filtering theory [8] that, under the stated condition, the closed linear hull in H of the predictors $\{E^{H^-} y(t);\ t \geq 0\}$ is finite-dimensional. But this is precisely $X_- := \bar{E}^{H^-} H^+$, and hence $\dim X_- < \infty$. Then, by symmetry, $X_+ := \bar{E}^{H^+} H^-$ is also finite dimensional, and consequently so is the frame space $H^{\square}$. Since all minimal splitting subspaces are contained in $H^{\square}$ [see (3.3)], the desired result follows. $\square$

COROLLARY 7.1. *Let the spectral density* Φ *be rational. Then all splitting subspaces contained in the frame space are finite dimensional.*

Now consider a finite dimensional regular Markovian splitting subspace with corresponding spectral factors $(W, \bar{W})$. Then, by Proposition 7.1, $K = \bar{W}^{-L} W$ is rational. Following Forney [9], we consider the vector space V_K of 2p-tuples over the field of rational functions, generated by the columns of the $2p \times p$ matrix $\tilde{K} = [I, K']'$. Then use the algorithm presented in [9] to find a minimal polynomial basis in V_K. Let n_i be the degree of the i:th element in this basis, i.e., the greatest degree of the 2p-polynomials in the i:th 2p-tuple. Then the minimality of the basis means that $n := n_1 + n_2 + \dots + n_p$ is as small as possible. Let $\tilde{Q} := [Q', \bar{Q}']'$ be the $2p \times p$ polynomial matrix formed by this basis. Then

$$K = \bar{Q}Q^{-1}, \tag{7.1}$$

where Q and $\bar{Q}$ are right coprime polynomial $p \times p$ matrices which are *column proper* [10], i.e., the high order coefficient matrices $[Q]_h$ and $[\bar{Q}]_h$ are

full rank ($[Q]_h$ is the constant matrix whose i:th column consists of the coefficients of s^{n_i} in the i:th column of Q.) The numbers $n_1, n_2, \ldots, n_p$ are called the *indices* of Q (or of $\bar{Q}$) [9; Main Theorem and Theorem 5].

LEMMA 7.1. *Suppose that* K *is given by (7.1). Let* $f \in (H_2^+K)^\perp$ *and define* $g = fQ$. *Then* $g \in R^p[s]$, *where* $R^p[s]$ *is the class of p-dimensional (row) vectors of polynomials with real coefficients.*

Proof. In view of (7.1), the representation

$$(H_2^+K)^\perp = \{f \in H_2^+ \mid fK^{-1} \in H_2^-\} \tag{7.2}$$

[7, p. 75] can be written in the form

$$(H_2^+K)^\perp = \{gQ^{-1} \in H_2^+ \mid g\bar{Q}^{-1} \in H_2^-\}. \tag{7.3}$$

Set $h := g\bar{Q}^{-1}$. Let $f \in (H_2^+K)^\perp$. Then, since $f \in H_2^+$, $g = fQ$ is analytic in the open right half plane. Similarly, since $h \in H_2^-$, $g = h\bar{Q}$ is analytic in the open left half plane. Also $f \in (H_2^+K)^\perp$ implies that f can be analytically extended to the imaginary axis [4; p. 270, Lemma 13.6], so therefore $g = fQ$ is analytic there too. Hence g is an entire function. We want to prove that $g \in R^p[s]$. To this end, first note that $k := \det K$ is a Blasche product, i.e., k is the product of relatively prime functions $k_i(s) := (s - s_i)^{n_i}/(s + \bar{s}_i)^{n_i}$ (where, for each function k_i, s_i is a complex number and n_i an integer), and that K is an inner divisor of kI [7; p. 70]. Then $H_2^+k \subset H_2^+K$ [7; p. 69] and $H_2^+k = \cap_i(H_2^+k_i)$ so that

$$(H_2^+K)^\perp \subset (H_2^+k)^\perp = \vee_i(H_2^+k_i)^\perp. \tag{7.4}$$

Therefore, if we could prove that, for each i, $(H_2^+k_i)^\perp$ is a space of rational vector functions, the same holds true for $(H_2^+K)^\perp$, and consequently the entire function g must be a vector polynomial. Since k_i is a scalar inner function, it suffices to prove that an arbitrary component of any vector

function in $(H_2^+k_i)^\perp$ is rational; hence, for the rest of the proof, it is no restriction to assume that H_2^+ is a Hardy space of *scalar* functions. Now, using Cauchy's formula, it can be shown [8; p.34] that

$$e_j(s) = \frac{1}{s + \bar{s}_i}\left(\frac{s - s_i}{s + \bar{s}_i}\right)^j \quad ; \; j = 0, 1, 2, \ldots$$

is an orthogonal basis of H_2^+. However, $e_j k_i = e_{j + n_i}$, and consequently $H_2^+k_i$ is the closed linear hull of $\{e_{n_i}, e_{n_i + 1}, \ldots\}$. Hence, $(H_2^+k_i)^\perp$ is the linear span of $\{e_o, e_1, \ldots, e_{n_1 - 1}\}$, which is a space of rational functions as required. □

PROPOSITION 7.3. *Let* X *be a finite dimensional regular Markovian splitting subspace and let* $(W,\overline{W})$ *be the corresponding pair of spectral factors. Then there is an* $m \times p$ *polynomial matrix* P *such that*

$$W = PQ^{-1} \quad \text{and} \quad \overline{W} = P\bar{Q}^{-1} \tag{7.5}$$

where Q *and* $\bar{Q}$ *are defined by the (right coprime and column proper) factorization (7.1) of (5.9). The function* P *has a left inverse.*

Proof. Equations (5.9) and (7.1) together yield $WQ = \overline{W}\bar{Q}$. Call this matrix function P; then equations (7.5) hold. Since W has a left inverse so does P. It remains to show that P is a matrix of polynomials. Since $WK^{-1} = \overline{W}$, (7.2) implies that the rows of W belong to $(H_2^+K)^\perp$. Hence, by Lemma 7.1, P is a polynomial matrix. □

LEMMA 7.2. *For* $i,j = 1, 2, \ldots, p$, *let* p_{ij} *and* q_{ij} *be polynomials such that* $p_{ij}/q_{ij} = (Q^{-1})_{ij}$. *Then, for each* i, $\deg p_{ij} - \deg q_{ij} \le - n_i$ *with equality for some* j.

Proof. By Cramer's rule, $[Q^{-1}(s)]_{ij} = (-1)^{i+j}\Delta_{ji}(s)/\Delta(s)$, where $\Delta := \det Q$ and Δ_{ji} is the determinant of the matrix obtained by deleting row j and

column i in Q. Hence, Δ_{ji} is a sum of products of one element from each of all columns of Q except the i:th, and consequently $\deg \Delta_{ji} \leq n - n_i$. Since $[Q]_h$ is full rank, for each i there is a j such that equality holds. In fact, in each column i of Q there is a row j such that deleting row j and column i in $[Q]_h$ leaves a nonsingular matrix. Hence in forming Δ_{ji} there is at least one product that contains only factors of highest degree. Since $\deg \Delta = n$, the lemma follows. □

THEOREM 7.1. *Let* K *be given by (7.1), and let* $n_1, n_2, \ldots, n_p$ *be the indices of* Q. *Then*

$$(H_2^+ K)^{\perp} = \Gamma_{(n_1, n_2, \ldots, n_p)} Q^{-1} \tag{7.6}$$

where $\Gamma_{(n_1, n_2, \ldots, n_p)}$ *is the set of all* $g \in R^p[s]$ *such that, for each* $i = 1, 2, \ldots, p$, $\deg g_i < n_i$.

Proof. From (7.3) it follows that the functions in $(H_2^+ K)^{\perp}$ are of the form gQ^{-1} where g is a row vector of polynomials (Lemma 7.1). It just remains to show that the functions g satisfying $gQ^{-1} \in H_2^+$ and $g\bar{Q}^{-1} \in H_2^-$ are precisely those in $\Gamma_{(n_1, n_2, \ldots, n_p)}$. But, since Q and $\bar{Q}$ have the same indices, this follows from Lemma 7.2. □

COROLLARY 7.2. *Let* X *be a finite dimensional regular Markovian splitting subspaces, let* $(W, \bar{W})$ *be the corresponding pair of spectral factors, and let* Q *and* $\bar{Q}$ *be column proper right coprime polynomial matrices satisfying*

$$\bar{Q} Q^{-1} = \bar{W}^{-L} W. \tag{7.7}$$

Then

$$X = \int_{-\infty}^{\infty} \Gamma_{(n_1, n_2, \ldots, n_p)}(i\omega) P^{-L}(i\omega) d\hat{y}(i\omega), \tag{7.8}$$

where P *is the polynomial matrix* $P = WQ$, $n_1, n_2, \ldots, n_p$ *are the indices of* Q, *and* Γ *is defined as in Theorem 7.1. The dimension of* X *is equal to* $n := n_1 + n_2 + \ldots + n_p$.

Proof. Theorems 5.2 and 7.1 imply that

$$X = \int_{-\infty}^{\infty} \Gamma_{(n_1, n_2, \ldots, n_p)}(i\omega)Q^{-1}(i\omega)d\hat{u}(i\omega). \tag{7.9}$$

Since $W = PQ^{-1}$, $d\hat{u} = QP^{-L}d\hat{y}$, and consequently (7.8) follows. By Proposition 7.3, P is a polynomial matrix. The statement concerning $\dim X$ is immediate. □

Next we proceed to find a basis in X. To this end, first define the $n \times p$ polynomial matrix

$$\Pi(s) = \mathrm{diag}\{\pi_{n_1}(s), \pi_{n_2}(s), \ldots, \pi_{n_p}(s)\} \tag{7.10}$$

where π_k is the k-dimensional column vector $\pi_k(s) := (1, s, s^2, \ldots, s^{k-1})'$ of powers of s, and $n_1, n_2, \ldots, n_p$ are the indices of Q. Then, since $[Q]_h$ is nonsingular, Q may be written

$$Q(s) = [Q]_h\{\mathrm{diag}(s^{n_1}, s^{n_2}, \ldots, s^{n_p}) + \tilde{A}\Pi(s)\}, \tag{7.11}$$

where $\tilde{A}$ is the $p \times n$-dimensional constant matrix of polynomial coefficients. Now, in view of (7.8), the stochastic vector x defined by

$$x = \int_{-\infty}^{\infty} \Pi(i\omega)P^{-L}(i\omega)d\hat{y}(i\omega) \tag{7.12}$$

is a basis in X.

To obtain a dynamical representation for x, define the shift matrix

$$J = \mathrm{diag}\{J_{n_1}, J_{n_2}, \ldots, J_{n_p}\}, \tag{7.13}$$

where J_k is the companion matrix with characteristic polynomial $\chi_{J_k}(s) = s^k$ [i.e., J_k is a $k \times k$ matrix with $(J_k)_{ij} = 1$ whenever $j = i + 1$ and zero otherwise], and the $p \times n$ matrix E with ones in positions $(\Sigma_{i=1}^{j} n_i, j)$ and zeros elsewhere.

LEMMA 7.3 [10]. *Let the* $n \times n$ *matrix* A *and the* $n \times p$ *matrix* B *be defined by*

$$\begin{cases} A = J - E\tilde{A} \\ B = E[Q]_h^{-1} \end{cases} \tag{7.14}$$

Then

$$\Pi(s)Q(s)^{-1} = (sI - A)^{-1}B. \tag{7.15}$$

Proof. By definition,

$$(sI - A)\Pi(s) = (sI - J)\Pi(s) + E\tilde{A}\Pi(s).$$

But it is not hard to see that

$$(sI - J)\Pi(s) = E\,\mathrm{diag}\{s^{n_1}, s^{n_2}, \ldots, s^{n_p}\}$$

and consequently $(sI - A)\Pi(s) = BQ(s)$. □

The pair (A,B) defined by (7.14) is the Luenburger *controllable canonical form*, and $n_1, n_2, \ldots, n_p$ are the controllability indices [9,10].

THEOREM 7.2. *Let* $\operatorname{rank} \Phi = p$. *Then to each regular Markovian splitting subspace* X *of dimension* $n < \infty$ *there corresponds a stochastic realization*

$$\begin{cases} dx = Axdt + Bdu & (7.16a) \\ y = Cx, & (7.16b) \end{cases}$$

where A, B *and* C *are constant matrices of dimensions* $n \times p$, $n \times p$ *and* $m \times n$ *respectively, such that, for all* $t \in R$, $x(t)$ *is a basis in* $U_t X$, $\{u(t)\ ;\ t \in R\}$ *is a Wiener process satisfying* $H^-(du) = S = H^- \vee X$, A *is a stability matrix,* (C,A) *is observable, and* (A,B) *is in controllable canonical form with controllability indices* $n_1, n_2, \ldots, n_p$. *The triplet* (A,B,C) *is determined in the following way: Let (7.1) be a column proper and right*

coprime factorization of the inner function K *corresponding to* X, *and let* $n_1, n_2, \ldots, n_p$ *be the indices of* Q. *Then* A *and* B *are defined by (7.14) and* C *as the coefficient matrix of* P, *i.e.*,

$$C\Pi(s) = P(s). \tag{7.17}$$

Proof. Define the n-dimensional process $\{x(t)\ ;\ t \in R\}$ with components $x_k(t) := U_t x_k$, $k = 1, 2, \ldots, n$, x being the stochastic vector (7.12). Then $x(t)$ is a basis in $X_t := U_t X$ for all $t \in R$, and, since $P^{-L} d\hat{y} = Q^{-1} d\hat{u}$,

$$x(t) = \int_{-\infty}^{\infty} e^{i\omega t}\Pi(i\omega)Q^{-1}(i\omega)d\hat{u}(i\omega), \tag{7.18}$$

where the Wiener process $\{u(t)\ ;\ t \in R\}$ is determined by (5.5) and Theorem 5.1. Now, inserting (7.15) into this and transforming to the time domain, we obtain

$$x(t) = \int_{-\infty}^{t} e^{A(t-\tau)}Bdu(\tau), \tag{7.19}$$

which is the integrated version of (7.16a). In fact, since det Q is the characteristic polynomial of A [9], all eigenvalues of A lie in the open left half plane, and, since $\tilde{Q}$ is a minimal basis in V_K (i.e., n is as small as possible), the pair (C,A) is observable. To establish (7.16b), first note that, in view of (3.3), $y(o) \in X$; hence, there is a matrix C such that $y(o) = Cx$. Applying the operator U_t to this, (7.16b) follows. It is clear from (7.12) that C is given by (7.17). □

Note that we can obtain a whole class of equivalent realizations (7.16) through transformations of the type

$$(A,B,C) \longmapsto (T^{-1}AT,\ T^{-1}B,\ CT),$$

where T is any nonsingular constant matrix.

The complete analysis of this section can be repeated with time reversed. Indeed, by replacing Q and $\bar{Q}$ in the derivation above, we obtain a realization (7.16) in which the matrix A has all its eigenvalues in the open right

half plane. To justify the exchange of Q and $\bar{Q}$, observe that (5.11) may also be written

$$X = \int (H_2^- K)^{\perp} d\hat{u},$$

where now $\perp$ denotes the orthogonal complement in H_2^-. Then the situation is completely symmetric.

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